2-d Gravity as a Limit of the SL(2,R) Black Hole

M. Alimohammadi [(a,b)] and F.Ardalan [(a,c,d)]

[a] *Institute for Studies in Theoretical Physics and Mathematics*

P.O.Box 19395-5746, Tehran, Iran

[b] *Physics Department, Tehran University , North Karegar ,*

Tehran, Iran

[c] *Physics Department, Sharif University of Technology*

P.O.Box 11365-9161, Tehran, Iran

[d] *Department of Physics, University of Cincinnati,*

Cincinnati, Ohio 45221, U.S.A.

Abstract

The transformation of the $SL(2,R)/U(1)$ black hole under a boost of the subgroup $U(1)$ is studied. It is found that the tachyon vertex operators of the black hole go into those of the $c=1$ conformal field theory coupled to gravity. The discrete states of the black hole also tend to the discrete states of the 2-d gravity theory. The fate of the extra discrete states of the black hole under boost are discussed.

1 Introduction

The relation between the two simple string theory models in two dimensions, the critical $U(1)$ gauged WZW $SL(2,R)$ model [1−4], and the noncritical string theory of a one dimensional matter field coupled to Liouville field, has attracted considerable attention [5−14].

In Ref.[1] it was argued that as it is not possible to remove one of the parameters of the two dimensional black hole in favour of the Liouville field in all the regions of the black hole geometry, the theory can not be regarded as a non-critical string theory of $c = 1$ matter coupled to gravity. In agreement with this result, Distler and Nelson[6] studied the BRST cohomology of the black hole and found that there are more discrete states in the black hole than in the Liouville theory. Also by looking at the behaviour of the states of the black hole near the horizon, it was found [12] that there are only a few states that do not diverge near the horizon and are therefore physical, the W_∞ states not being among them.

On the other hand using the free field realization of $SL(2,R)/U(1)$ and the true BRST charge in the black hole, it was shown that as far as the energy-momentum tensor is concerened, the model is identical to 2d gravity [8]. It was then claimed that the extra states that appear in the $SL(2,R)/U(1)$, are BRST exact and therefore the spectrum of both theories are the same. It has been argued that [13] there are null states in the black hole which lead to even more discrete states than in Ref.[6]. Yet in a different approach the number of discrete states of the black hole come out even less than those of 2-d gravity [12]. On the other hand in the context of matrix models, it has been shown that the 2 dimensional black hole theory and the 2 dimensional gravity theory are closely related [14]. Therefore the relation between the two theories warrants further investigation.

In this work we will study the relation by a different method, i.e., by gauging the

SL(2,R) by a nilpotent subgroup and looking at the behaviour of the black hole theory $SL2,R)/U(1)$ when the $U(1)$ tends towards this nilpotent subgroup, which we call $E(1)$. When the $U(1)$ subgroup of the $SL(2,R)/U(1)$ black hole is substituted with the subgroup $E(1)$, it is found [15,16] that the resulting theory is nothing but the one dimensional Liouville field theory with zero cosmological constant [18]. The same result can of course be obtained by boosting the original $U(1)$ subgroup and letting the boost parameter go to infinity. This reduction of the degrees of freedom from two to one, has been studied and is understood to be a consequence of an enlarged symmetry [17]. In the effort to understand the details of the effect of the boost on the black hole theory, the effective Lagrangian of the boosted theory was studied in the limit of large but finite boost parameters, and found to resemble the corresponding quantity in the theory of a c=1 conformal matter coupled to Liouville field[15,16].

In this paper we have pursued this line of investigation in more detail and have found that not only the action and the tachyon of the black hole tend to those of 2-d gravity, but also the discrete states of the black hole tend to the discrete states of the 2-d gravity. The identification of the quantum numbers of the former theory with the momenta of the latter[6], will then appear as a natural consequence of the boost transformation. The transformation also explains the disappearance of the extra discrete states which occur in black hole and not in the 2-d gravity.

In section 2 we will review the results of Ref.[16] for E(1) gauging and discuss the free field representation of the nilpotent gauged WZW model of $SL(2,R)$ and show that the stress tensor of this model is the same as in the Liouville theory. In section 3 we will study the limit of the primary fields of $SL(2,R)/U_t(1)$ at $t \to \infty$, where t is the boost parameter, and show that the primary fields in the regions V and III of Ref.[1] lead to the vertex operators of the c=1 theory coupled to Liouville. In section 4 we study the

operator aspects of the the boost transformation and in section 5 we will take up the discrete states.

2 Gauging SL(2,R) by its Nilpotent Subgroup

Let us take the following parametrization for elements $g \in SL(2,R)$,

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \qquad ab + uv = 1. \tag{1}$$

We consider the nilpotent subgroup E(1) of SL(2,R) generated by $\sigma = \sigma_3 + i\sigma_2$ and use the axial gauge freedom ($g \to hgh$) to fix the gauge by the following condition :

$$a + b = 0, \tag{2}$$

which is valid in region V . Then using the parameters:

$$\begin{aligned} x &= \frac{1}{2}(u - v) \\ e^{\varphi'} &= \frac{1}{4}(u + v - 2a), \end{aligned} \tag{3}$$

we find the following effective action [16]

$$I_{eff} = \frac{k}{4\pi}\int d^2\sigma\sqrt{h}h^{ij}\partial_i\varphi'\partial_j\varphi' + \frac{1}{4\pi}\int d^2\sigma\sqrt{h}R^{(2)}\varphi'. \tag{4}$$

In the above equation h^{ij} is the two dimensional metric and $R^{(2)}$ is the curvature of the world-sheet . This one-dimensional action is nothing but the Liouville action . We will now show this equivalence of $SL(2,R)/E(1)$ and Liouville theory at the level of stress tensor . As is well known, if one uses the Gauss decomposition to represent the group elements of $SL(2,R)$, the following representations for the currents of $SL(2,R)_k$ in terms of free fields β ,γ and ϕ can be obtained [19]:

$$J_+ = \beta$$

$$J_- = \beta\gamma^2 + \sqrt{2k'}\gamma\partial\phi + k\partial\gamma \quad , \quad k' = k-2 \tag{5}$$

$$J_3 = -\beta\gamma - \sqrt{\frac{k'}{2}}\partial\phi$$

where β and γ are the commuting ghost fields with dimensions $h = 1, 0$ and with OPE's $\beta(z)\gamma(w) \sim \frac{1}{z-w}$ and $\phi(z)\phi(w) \sim -\lg(z-w)$. Then using Sugawara construction, the stress tensor of $SL(2,R)_k$ becomes:

$$T_{SL(2,R)}(z) = \beta\partial\gamma - \frac{1}{2}(\partial\phi)^2 - \frac{1}{\sqrt{2k'}}\partial^2\phi \tag{6}$$

Now we want to gauge away the nilpotent subgroup of $SL(2,R)$, i.e. J_+, by using the BRST method. As $J_+(z)J_+(w)$ is regular, there is no need to introduce a gauge field (auxiliary field) for constructing the BRST charge (Q_+), and hence there is no need to introduce ghosts to fix the gauge field. In this way we arrive at the following expression for the BRST charge of the nilpotent subgroup of $SL(2,R)$:

$$Q_+ = \oint dz J_+(z) \tag{7}$$

which satisfies,

$$Q_+^2 = 0$$

. As we do not introduce the gauge field, so it has no contribution in $T(z)$ and therefore

$$T_{SL(2,R)/E(1)} = \beta\partial\gamma - \frac{1}{2}(\partial\phi)^2 - \frac{1}{\sqrt{2k'}}\partial^2\phi \tag{8}$$

But there are terms in Eq.(8) which are BRST exact and must be subtracted from the stress tensor. It can be easily shown that:

$$\beta\partial\gamma = \partial(\gamma\beta) - \gamma\partial\beta$$

$$= \partial(\gamma\beta) - \{Q_+, \frac{1}{2}\partial\beta\gamma^2\}$$

$$= \{Q_+, \partial(\frac{1}{2}\beta\gamma^2)\} - \{Q_+, \frac{1}{2}\partial\beta\gamma^2\}$$

Thus up to a BRST exact term and at the level $k = \frac{9}{4}$, we find that:

$$T_{SL(2,R)/E(1)} = -\frac{1}{2}(\partial\phi)^2 - \sqrt{2}\partial^2\phi \tag{9}$$

The above equation is exactly the Liouville action at zero cosmological constant.The same result can be obtained if we consider the stress tensor of $SL(2,R)/U_t(1)$ and look at its behaviour at $t \to \infty$.

3 Primary Fields

The primary fields of WZW model are defined via its matrix elements. In the case of $SL(2,R)$ gauged by σ_3, the vertex operator in the region I is [3]:

$$V_\lambda^\omega = < \lambda, \omega | g(y,\tau) | \lambda, -\omega > \tag{10}$$

where λ defines the spin of $SL(2,R)$ representation, ω is the eigenvalue of σ_3 and y and τ are defined by:

$$y = uv, \qquad e^{2\tau} = -\frac{u}{v}. \tag{11}$$

In this region the suitable gauge condition is $a - b = 0$. There are four different vertex operators in the region I which have different behaviours near the horizon and infinity, among which one, denoted U_λ^ω, can be naturally extended to the region III,[3] :

$$U_\lambda^\omega = e^{-2i\omega\tau} F_\omega^\lambda(y) = e^{-2i\omega\tau}(-y)^{-i\omega} B(\nu_+, \bar{\nu}_-), F(\nu_+, \bar{\nu}_-, 1 - 2i\omega, y) \tag{12}$$

where $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, F is the hypergeometric function ${}_2F_1$, and

$$\nu_\pm = \frac{1}{2} - i(\lambda \pm \omega) \tag{13}$$

However, it is convenient to work with the boosted group element g , in Eq.(10), rather than using the state corresponding to the $U_t(1)$. We therefore have,

$$U_\lambda^\omega(t) =< \lambda,\omega|g(y_{-t},\tau_{-t})|\lambda,-\omega > \tag{14}$$

where

$$g_{-t} = e^{\frac{t}{2}\sigma_1} g e^{-\frac{t}{2}\sigma_1} \tag{15}$$

If the eigenvalues of $\sigma = \sigma_3 + i\sigma_2$ are denoted by χ

$$\sigma|\lambda,\chi >= \chi|\lambda,\chi >$$

then as $\sigma_3^t \stackrel{t\to\infty}{\longrightarrow} e^t\sigma$ the states $|\lambda,\omega>_t$ must tend to $|\lambda,\chi>$ as $t\to\infty$. As a result,

$$\omega = \chi e^t \tag{16}$$

and

$$\nu_\pm = \frac{1}{2} - i(\lambda \pm \chi e^t) \tag{17}$$

To find the limit of the vertex operators , we will express the hypergeometric function in terms of the associated Legendre function of the 2^{nd} kind $Q_\mu^\nu(z)$ [20], and obtain,

$$U_\lambda^\chi(t) \stackrel{t\to\infty}{\longrightarrow} e^{-4i\chi\frac{u+v}{u-v}} Q_\nu^0(1+\frac{8}{(u-v)^2})$$

which when $|\nu|\to\infty$, reduces to:

$$U_\lambda^\chi(t) \to e^{-4i\chi\frac{u+v}{u-v}}\sqrt{\frac{\pi}{2}}(w^2-1)^{\frac{-1}{4}}(w-\sqrt{w^2-1})^{-\frac{i}{2}\chi e^t} \tag{18}$$

where $w = 1+\frac{8}{(u-v)^2}$. As expected, there is no connection to 2-d gravity in the region I.

The vertex operators in the region V are [3] :

$$W_\omega^\lambda(y,\tau) = e^{-2i\omega\tau}y^{-i\omega}F(\nu_+,\bar{\nu}_-,1,1-y) \tag{19}$$

where $y = uv$ and $\tau = \frac{1}{2}\ln(u/v)$ and the gauge condition is $a+b=0$. As this function has no singularity when crossing the singularity at $y=1$, it can be trivially continued

to region III. In the same way as discussed in the previous section, the corresponding vertex operator of $SL(2,R)/U_t(1)$ in this region can be recovered from Eq.(19) by simply transforming $y \rightarrow y_{-t}$, $\tau \rightarrow \tau_{-t}$ and taking the gauge condition $a_{-t} + b_{-t} = 0$ .After some algbra, using a similar set of identities as in the case of region I, we obtain,

$$W_\chi^\lambda(t) \stackrel{t\rightarrow\infty}{\longrightarrow} \frac{1}{\pi} e^{-2ix\chi e^{-\varphi'}} e^{\pi\chi e^t} e^{-(t+\varphi')} K_{2i\lambda}(2\chi e^{-\varphi'}) \tag{20}$$

Fortunately the dependence of W_χ^λ on t is such that we can absorb it consistently in φ', and therefore if we define $\varphi = \varphi' + t$ and $xe^{-\varphi} = X$ and use Eq.(16), we will finally obtain,

$$W_\omega^\lambda(t) \stackrel{t\rightarrow\infty}{\longrightarrow} \frac{e^{\pi\omega}}{\pi} e^{-2i\omega X} e^{-\varphi} K_{2i\lambda}(2\omega e^{-\varphi}) \tag{21}$$

The Eq.(21) is exactly the vertex operator of c=1 coupled to 2-d gravity with non-zero cosmological constant [21]. This equivalence is even clearer when the vertex operator is considered on-shell, that is when $\lambda = \pm\omega/3$ [3]. Eq.(20) also shows that the eigenvalue of σ plays the role of the cosmological constant $\chi = \sqrt{\mu}$ [3]. When $\chi \rightarrow 0$, we obtain,

$$W_\omega^\lambda(t) \stackrel{t\rightarrow\infty}{\longrightarrow} e^{-(\varphi'+t)} e^{-ix\chi e^{-\varphi'}} \{A e^{2i\lambda(\varphi'+t)} + c.c.\} \tag{22}$$

where:

$$A = \frac{\Gamma(2i\lambda)}{\Gamma(\frac{1}{2} + i\lambda + i\omega)\Gamma(\frac{1}{2} + i\lambda - i\omega)}$$

As in the previous case, if we define $\varphi = \varphi' + t$ and $X = xe^{-\varphi}$ we will arrive at the following expression for the primary fields

$$W_\omega^\lambda \rightarrow e^{-\varphi} e^{-2i\omega X} \{A e^{2i\lambda\varphi} + A^* e^{-2i\lambda\varphi}\} \tag{23}$$

However, this is nothing but the vertex operator of c=1 plus Liouville at zero cosmological constant, of course after applying the on-shell condition. The scattering matrix is A/A^*. Therefore χ plays exactly the role of the cosmological constant and the boosted black hole in region V is equivalent to 2-d gravity. At $t = \infty$ where $\chi = \omega e^{-t} = 0$, Eq.(22) becomes:

$$W_\omega^\lambda(t = \infty) = A e^{2i(\lambda-1)\varphi} + c.c. \tag{24}$$

which is the expression for the vertex operator of pure gravity, Liouville theory, as expected: as $t \to \infty$, $\sigma_3^t \to \sigma$ and we expect the theory to reduce to $SL(2,R)/E(1)$, which is the Liouville theory.

4 Limit of the Operators

In the region V where the gauge condition is $a + b = 0$,the effective action is [1] :

$$I_{eff} = -\frac{k}{4\pi}\int \frac{\partial u\bar{\partial}v + \partial v\bar{\partial}u}{1-uv} d^2z + \frac{1}{4\pi}\int d^2\sigma\sqrt{h}R^{(2)} ln(1-uv). \tag{25}$$

If we boost this action and keep the next to leading order terms in $\epsilon = e^{-t}$, we will find :

$$\begin{aligned} I_{eff} = \frac{k}{4\pi}\int d^2\sigma\sqrt{h}h^{ij}[\partial_i\varphi\partial_j\varphi - (\partial_i X\partial_j X + X\partial_i\varphi\partial_j X)] \\ + \frac{1}{2\pi}\int d^2\sigma\sqrt{h}R^{(2)}(\varphi - \frac{1}{4}X^2), \end{aligned} \tag{26}$$

where $X = xe^{-\varphi}$ and $\varphi = \varphi' + t$, which resembles the theory of a matter field X coupled to a Liouville field φ, including an interaction term [5]; which we may ignore if we assume that X is small, because of its ϵ dependence, and if ∂X is comparable to $\partial\varphi$. Note that if we redefine the Liouville field as,

$$\Phi = \varphi - \frac{1}{4}X^2, \tag{27}$$

we obtain,

$$I_{eff} = \frac{k}{4\pi}\int d^2\sigma\sqrt{h}h^{ij}(\partial_i\Phi\partial_j\Phi - \partial_i X\partial_j X) + \frac{1}{2\pi}\int d^2\sigma\sqrt{h}R^{(2)}\Phi, \tag{28}$$

which indeed is the standard 2-d gravity action. Observe that the correction to φ is only second order in ϵ and will not affect the results for the tachyon vertex operators of previous section. To go further we need to know the operator properties of the fields in the boosted theory.

To investigate the OPE's of these fields, it is necessary to consider the free field representation of SL(2,R). As it is known, if we use the Gauss decomposition to parametrize the SL(2,R) group elements , that is :

$$g = e^{\gamma\sigma_+} e^{\phi'\sigma_3} e^{\chi\sigma_-} = \begin{pmatrix} e^{\phi'} + \chi\gamma e^{-\phi'} & \gamma e^{-\phi'} \\ \chi e^{-\phi'} & e^{-\phi'} \end{pmatrix} \tag{29}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, then the currents $J = J_i\sigma_i = -k(\partial g)g^{-1}$ reduce to the representation (5) with definition of ϕ and β as :

$$\begin{aligned} \phi &= -k\sqrt{\frac{2}{k'}}\phi' \\ \beta &= -k\partial\chi e^{-2\phi'}. \end{aligned} \tag{30}$$

Boosting the representation (29) and imposing the gauge condition, results in the following relations for the leading terms of β_{-t}, γ_{-t} and ϕ_{-t} :

$$\begin{aligned} \beta_{-t} &= k(x\partial\varphi - \partial x)e^{\varphi} = -k\partial X e^{2\varphi} \\ \gamma_{-t} &= 1 + xe^{-\varphi} = 1 + X \\ \phi_{-t} &= k\sqrt{\frac{2}{k'}}\varphi \end{aligned} \tag{31}$$

Now the OPE's of the above fields are :

$$\begin{aligned} &\beta_{-t}(z)\gamma_{-t}(w) \sim \frac{1}{z-w} \quad , \quad \phi_{-t}(z)\phi_{-t}(w) \sim -lg(z-w) \\ &\beta_{-t}(z)\phi_{-t}(w) \sim \gamma_{-t}(z)\phi_{-t}(w) \sim regular \end{aligned} \tag{32}$$

therefore using the Eq.(31), the following OPE's are dictated for X and φ fields :

$$\begin{aligned} \varphi(z)\varphi(w) &\sim -\frac{k'}{2k^2}lg(z-w) \\ \varphi(z)X(w) &\sim regular \\ < X(z)X(w) > &= -\frac{1}{2k}lg(z-w). \end{aligned} \tag{33}$$

The above equations show that the interpretation of φ and X as the Liouville and the c=1 bosonic fields, which commute with each other, is permitted.

5 Discrete States

The discrete states of the $SL(2,R)/U(1)$ black hole were found in reference [6] using the parafermionic modules $V_{j,m}$ built on the states $U_{j,m}$ of the discrete irreducible representations of $SL(2,R)$, where j and m are the usual angular momentum lables of $SL(2,R)$. It was found that aside from the propagating tachyon states with

$$m = \pm 3(j+1)/2,$$

there are three sets of discrete states labled as D ,C and $\hat{D}$ with

$$m = \pm\frac{3}{8}(2s-4r-1) \quad , \quad j = \frac{1}{8}(2s+4r-5)$$

for $\hat{D}$ states; and

$$m = \pm\frac{3}{4}(s-2r+1) \quad , \quad j = \frac{1}{4}(s+2r-3)$$

for D states; and

$$m = \frac{3}{2}(s-r) \quad , \quad j = \frac{1}{2}(s+r-1)$$

for the C states, where s and r are positive integers. From the form of the discrete states of the 2-d gravity theory,

$$p_x = \sqrt{2}(p-q) \quad , \quad p_\varphi = \sqrt{2}(p+q-1),$$

with p and q positive integers , it was suggested that the sets D and C correspond to the 2-d gravity discrete states with the identification

$$p_x = 2\sqrt{2}m/3 \quad , \quad p_\varphi = 2\sqrt{2}j; \tag{34}$$

and that the $\hat{D}$ did not correspond to any 2-d gravity states. These extra states will of course not appear if a free field representation of $SL(2,R)/U(1)$ is used, as the operator structure and the energy momentum tensor of the theories are not distinguishable in this

representation. However, once the Kac-Moody Verma modules are usd, the extra states are inescapable. Below, we will see that performing a boost transformation on the states to the black hole will remove the $\hat{D}$ states and reduce the spectrum to that of the 2-d gravity.

To begin with, recall that the states $\hat{D}$ and D are related by the screening operators,

$$S^{\pm} = \oint dz e^{\sqrt{k'/2}\phi^1 \pm i\sqrt{k/2}\phi^2},$$

in terms of the free fields ϕ^1 and ϕ^2. In fact $\hat{D} = \ker\ (S)$ and $D = V/\hat{D}$, and the modules V are generated on the SL(2,R) base representation operators,

$$U_{j,m} = e^{j\sqrt{2/k'}\phi^1 + m\sqrt{2/k}(i\phi^2+\phi^3)},$$

where ϕ^3 is the $U(1)$ free field. We now apply our boosting map on these operators and obtain,

$$\begin{aligned} U_{j,m} &\to const.e^{-2j\frac{k}{k'}\varphi' - 2(j+m)X}, \\ S^{+} &\to const. \oint dz e^{-k\varphi' - 2(k-1)X}, \\ S^{-} &\to const. \oint dz e^{-k\varphi' - 2x}. \end{aligned} \tag{35}$$

From these equations, as we have interpreted φ to be the Liouville field, considering the normalisation Eq.(33) of φ, we see that

$$p_{\varphi} = 2\sqrt{2}j, \tag{36}$$

as suggested in Ref.[6]. The corresponding relation for p_x,

$$p_x = 2\sqrt{2}(m+j)/3, \tag{37}$$

has the coefficient suggested in Ref.[6], and m substituted for $m + j$.

Next, taking the OPE's of the screening operators with the Verma module operators, we see that in the approximation we are considering, the screening operators fail to be

well defined, indicating that the states $\hat{D}$ are removed from the spectrum. To confirm the result above we have calculated the commutator of the limit of the screening operators with that of the Virasoro operator L_0, and they also fail to vanish in the first order approximation theory. We conclude that in the large boost limit of the black hole theory, the extra discrete states disappear and the spectrum becomes that of the 2-d gravity.

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